Mock Exam Analysis on Manifolds

Note: This exam consists of problems. Usage of the theory and examples from the lecture notes is allowed. Give a precise reference to the theory and/or exercises you use for solving the problems.
You get 10 points for free.

Problem 1. (5 + 10 + 10 = 25 points) *This is Exercise 5.4.3, before using it as a homework I planned it as an exam exercise as follows.*

For this problem you cannot use the results in Chapter 8.2 and 8.3. We are going to show that line integrals are well defined and generalize the fundamental theorem of calculus.

Let M be a smooth manifold, $\gamma : I = [a,b] \subset \mathbb{R} \to M$ a smooth curve and $\omega \in \mathfrak{X}^*(M)$ a 1-form. Show the following properties.

(a) Show that for line integrals

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t))\, dt. \tag{1}$$

(b) Let $J \subset \mathbb{R}$ be an open interval and $F : J \to I$ an orientation-preserving diffeomorphism. Show that

$$\int_{F^*\gamma} \omega = \int_\gamma \omega. \tag{2}$$

Hint: use the chain rule to get $(F^*\gamma)'(t) = \gamma'(F(t))F'(t)$ *and then apply* (1).

(c) Let $f \in C^\infty(M)$. Prove the fundamental theorem of calculus:

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)). \tag{3}$$

Hint: reduce the formula to the usual fundamental theorem of calculus on $\mathbb{R}$.

Problem 2. (7 + 10 + 10 + 5 = 32 points)
The goal of this exercise is to show that the three sphere is parallelizable without using the theory of Lie groups.
Consider $\mathbb{S}^3 = \{(x,y,z,w) \in \mathbb{R}^4 \mid x^2+y^2+z^2+w^2 = 1\} \subset \mathbb{R}^4$.

(a) Show that $T_p\mathbb{S}^3$ as a subspace of $T_p\mathbb{R}^4$ is identified with $p^\perp := \{q \in \mathbb{R}^4 \mid p\cdot q = 0\}$.

(b) Show that

$$X = w\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} - x\frac{\partial}{\partial w}$$

is tangent to $\mathbb{S}^3$.

(c) Find another vector field Y, given by a similar formula, that is also tangent to $\mathbb{S}^3$ and such that X, Y and $Z := [X,Y]$ span the tangent space $T_p\mathbb{S}^3$ for all $p \in \mathbb{S}^3$.

(d) Show that $T\mathbb{S}^3$ is trivializable.

Problem 3. (8 + 10 + 5 + 10 = 33 points) Let V a vector space of dimension k. A symplectic form on V is an element $\omega \in \Lambda^2(V)$ which is non-degenerate in the sense that $\iota_v(\omega) = 0$ if and only if $v = 0$.
A *symplectic manifold* is a smooth manifold M equipped with a closed differential 2-form ω such that ω_q is a symplectic form on T_qM for every $q \in M$.

(a) Prove that if a symplectic form exists, then $k = 2n$ for some $n \in \mathbb{N}$, i.e., it must be an even number.

(b) Let M be a smooth manifold. Define a 1-form $\eta \in \Omega^1(T^*M)$ on the cotangent bundle of M as

$$\lambda_{(q,p)} = d\pi^*_{(q,p)}p, \quad q \in M,\ p \in T^*_qM, \tag{4}$$

where $\pi : T^*M \to M$ is the projection to the base. If (x^i, ξ_i) denote the local coordinates induced by a chart of M on T^*M, show that $\lambda_{(x,\xi)} = \xi_i dx^i$.

(c) Show that $\omega := d\lambda$ is a symplectic form on T^*M, that is, every cotangent bundle is a symplectic manifold.

(d) Use $\omega = d\lambda$ to show that T^*M is orientable.